INFOMGP - GAME PHYSICS

EXERCISES LECTURE 4

EXERCISE 4.1

Calculate or determine the mass and the center of mass for the following objects.

i. A box with size (40 cm, 80 cm, 50 cm) centered at (-20 cm, 40 cm, 20 cm) with uniform density of 1400 kg/m^3.

ii. A cylinder shell with inner radius 1 meter and outer radius 2 meters, aligned along the x-axis with length 40 cm and its center at (20 cm, 20 cm, 20 cm) with uniform density of 300 kg/m^3.

iii. The composite object consisting of the two objects above (i. and ii.).

i.

The mass is the volume multiplied by the density.
Here we have $M_{box} = w \times h \times d \times \rho = 0.4 \times 0.8 \times 0.5 \times 1400 = 224$ kg
The center of mass is the center of the object, therefore $COM_{box} = (-0.2\,, 0.4\,, 0.2)$

ii.

Again the mass is the volume multiplied by the density. The volume of the cylinder shell is the volume of the big cylinder minus the volume of the small cylinder.
$M_{cylinder} = l \times \pi \times ({r_2}^2 - {r_1}^2) \times \rho = 0.4 \times \pi \times (4 - 1) \times 300 = 1131$ kg
The center of mass of the object is again the center of the object itself $COM_{cylinder} = (0.2\,, 0.2\,, 0.2)$

iii.

The mass of the composite object is the sum of the two objects:
$M_{composite} = M_{box} + M_{cylinder} = 224 + 1131 = 1355$ kg
The center of mass of the composite object can be calculated by a weighted sum:

$$COM_{composite} = \frac{M_{box} \times COM_{box} + M_{cylinder} \times COM_{cylinder}}{M_{composite}}$$

$$COM_{composite} = \frac{224\begin{pmatrix}-0.2\\0.4\\0.2\end{pmatrix} + 1131\begin{pmatrix}0.2\\0.2\\0.2\end{pmatrix}}{1355} = \begin{pmatrix}0.14\\0.23\\0.2\end{pmatrix}$$

EXERCISE 4.2

Show that the moment of inertia in the z-direction I_{zz} for a box with mass m and dimensions (w, h, d) is given by $I_{zz} = \frac{1}{12} m(w^2 + h^2)$.

$$I_{zz} = \int (x^2 + y^2)\, dm = \rho \int (x^2 + y^2)\, dV = \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} (x^2 + y^2)\, dx\, dy\, dz$$

$$I_{zz} = \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{1}{3}x^3 + y^2 x\right]_{-\frac{w}{2}}^{\frac{w}{2}} dy\, dz = \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{1}{12}w^3 + y^2 w\right) dy\, dz$$

$$I_{zz} = \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[\frac{1}{12}w^3 y + \frac{1}{3}y^3 w\right]_{-\frac{h}{2}}^{\frac{h}{2}} dz = \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} \left(\frac{1}{12}w^3 h + \frac{1}{12}h^3 w\right) dz = \rho \left[\left(\frac{1}{12}w^3 h + \frac{1}{12}h^3 w\right) z\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$I_{zz} = \rho \left(\frac{1}{12}w^3 hd + \frac{1}{12}h^3 wd\right) = \rho \left(w\, h\, d \frac{1}{12}(w^2 + h^2)\right) = \frac{1}{12} m(w^2 + h^2)$$

EXERCISE 4.3

Show that the moment of inertia in the z-direction I_{zz} for a cylinder oriented along the z-axis with mass m and radius r is given by $I_{zz} = \frac{1}{2} m r^2$.

A straightforward solution similar to the previous exercise is very difficult. So we will integrate using a small cylindrical ring instead of a small box. This ring has radius r, thickness dz and 'incremental size' dr. The moment of inertia becomes:

$$I_{zz} = \int (x^2 + y^2)\, dm = \rho \int (x^2 + y^2)\, dV = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{0}^{r} (x^2 + y^2) \times 2\pi r\, dr\, dz = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{0}^{r} r^2 \times 2\pi r\, dr\, dz$$

$$= 2\pi\rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{0}^{r} r^3\, dr\, dz = 2\pi\rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \left[\frac{1}{4}r^4\right]_0^r dz = \frac{1}{4} 2\pi\rho \int_{-\frac{l}{2}}^{\frac{l}{2}} r^4\, dz = \frac{1}{2}\pi\rho [r^4 z]_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{1}{2}\pi\rho r^4 l$$

$$= \frac{1}{2} m r^2$$

(as $m = \rho\, l\, \pi\, r^2$)

EXERCISE 4.4

Calculate the moment of inertia along the x-axis I_{xx} for the three objects of exercise 4.1. Please note that the moment of inertia of a cylinder shell along the x-axis is given by $I_{xx} = \frac{1}{2}m({r_1}^2 + {r_2}^2)$.

i.

We have a moment of inertia for the box $I_{xx} = \frac{1}{12}m(h^2 + d^2)$ where $h = 0.8$ and $d = 0.5$. Thus we have a moment of $I_{xx} = \frac{1}{12}224(0.8^2 + 0.5^2)$.

ii.

We have a moment of inertia for the cylinder shell $I_{xx} = \frac{1}{2}m({r_1}^2 + {r_2}^2)$ where $r_1 = 1$ and $r_2 = 2$. Thus we have a moment of $I_{xx} = \frac{1}{2}1131(1^2 + 2^2)$.

iii.

The moment of inertia of the composite object is the sum of the two moments.

However, since the box is not rotating around its own center of mass we have to use the parallel axis theorem to calculate the final I_{xx} where ${d_x}^2 = (\Delta y^2 + \Delta z^2)$. We then have:
$I_{xx} = \frac{1}{12}224(0.8^2 + 0.5^2) + 244(0.4^2 + 0.2^2) = 61.4$ kg m^2

Since the cylinder shell is not rotating around its own center of mass either, we have to use the parallel axis theorem to calculate the final I_{xx} where ${d_x}^2 = (\Delta y^2 + \Delta z^2)$. We then have:
$I_{xx} = \frac{1}{2}1131(1^2 + 2^2) + m{d_x}^2 = \frac{1}{2}1131(1^2 + 2^2) + 1131(0.2^2 + 0.2^2) = 2917.98$ kg m^2

Finally, we have:
$I_{xx} = I_{xx,box} + I_{xx,cylinder} = 61.4 + 2918 = 2979.38$ kg m^2

EXERCISE 4.5

Given a ring with radius R of n cylinders, each having a mass of M/n, and a radius r ($\ll R$). Determine the I_{zz} for the entire system of n cylinders. Remember that I_{zz} for a cylinder is $\frac{1}{2}mr^2$. What happens if n becomes larger and larger?

The given moment of inertia is around the own axis of the cylinder. Therefore we have to use the parallel axis theorem to find the moment of inertia of the cylinders in the entire system:

$$I_{zz} = \frac{1}{2}mr^2 + m{d_z}^2 = \frac{1}{2}mr^2 + mR^2$$

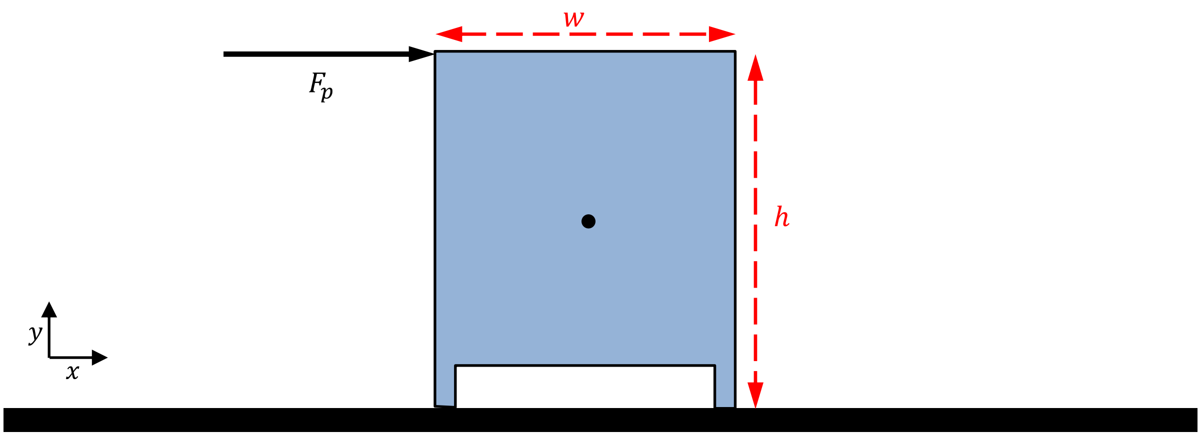
To calculate the I_{zz} for the entire system we have to sum up the moments of inertia over all cylinders:

$$I_{zz} = \sum_{i=1}^{n} \frac{1}{2} m_i r^2 + m_i R^2 = \frac{1}{2} M r^2 + M R^2$$

Since $r \ll R$, this can be simplified by $I_{zz} = MR^2$ which is the moment of inertia of a hollow cylinder. Therefore the surface of a hollow cylinder can be approximated by an infinite number of infinitely small cylinders.

EXERCISE 4.6

Analyze the forces and torques on the box. What force F_p is necessary to tip the box over?



First we identify and quantify the forces. We have the pushing force $F_p = \begin{pmatrix} f_p \\ 0 \\ 0 \end{pmatrix}$, the gravity force $F_g = \begin{pmatrix} 0 \\ -f_g \\ 0 \end{pmatrix}$ and the normal force $F_n = \begin{pmatrix} 0 \\ f_n \\ 0 \end{pmatrix}$. This last force only exists on the right leg of the box when it starts to tip over. Second we identify the torques in the system. The point of rotation of the system is the right leg position. Therefore, the moment arm r for the torque τ_p due to the pushing force is given by $r = \begin{pmatrix} -w \\ h \\ 0 \end{pmatrix}$. Thus the torque is calculated by:

$$\tau_p = r \times F_p = \begin{pmatrix} -w \\ h \\ 0 \end{pmatrix} \times \begin{pmatrix} f_p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -h\, f_p \end{pmatrix}$$

The moment arm r for the torque τ_g due to the gravity force is given by $r = \begin{pmatrix} -w/2 \\ h/2 \\ 0 \end{pmatrix}$. Thus the torque is calculated by:

$$\tau_g = r \times F_g = \begin{pmatrix} -w/2 \\ h/2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -f_g \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{w}{2} f_g \end{pmatrix}$$

The moment arm for the torque due to the normal force is zero, so there is no torque due to the normal force.

For the box to tip over, the angular acceleration has to be positive, *i.e.* $\alpha > 0$. Since the sum of all torques is equal to the moment of inertia times the angular acceleration, we have for that system:

$$\tau_p + \tau_g = I \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ -h\, f_p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{w}{2}\, f_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

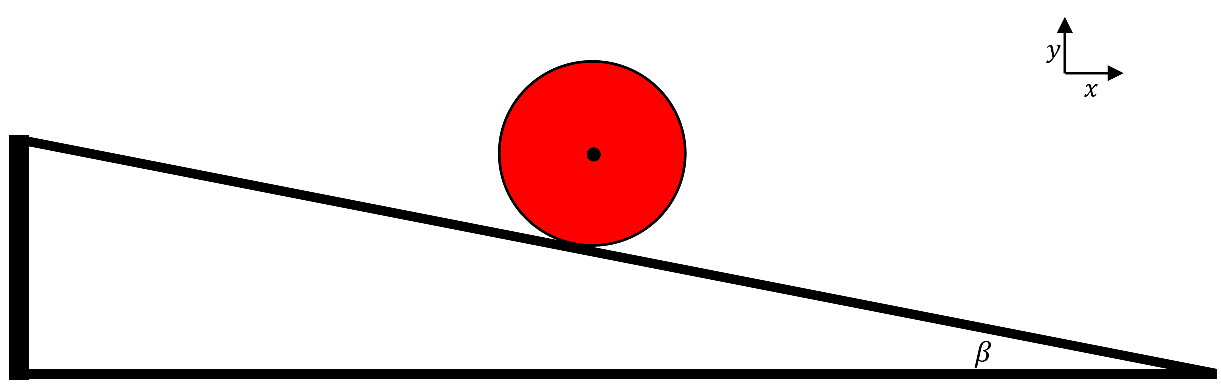
Therefore we have

$$h\, f_p = \frac{w}{2}\, f_g \Leftrightarrow f_p = \frac{w}{2h}\, f_g$$

So finally the necessary pushing force to tip over the box is $F_p = \begin{pmatrix} \frac{w}{2h}\, f_g \\ 0 \\ 0 \end{pmatrix}$.

EXERCISE 4.7

Analyze the forces and torques on the rolling cylinder of radius r. What is its linear acceleration?



First we identify and quantify the forces. We have the gravity force $F_g = \begin{pmatrix} 0 \\ -f_g \\ 0 \end{pmatrix}$ and the normal force F_n which is perpendicular to the slope. The last force is the friction force F_f which is oriented parallel to the slope. As these last two forces are oriented in a direction related to the slope, it is convenient to work on the linear forces in that direction. Thus the gravity force needs to be transformed into the scalar values $F_g^{\perp}$ and $F_g^{\parallel}$.
The cylinder rotates around its center of mass. The center of mass also moves linearly parallel to the slope. Thus the cylinder has an angular acceleration α and a linear acceleration a. Since the cylinder

does not slide, these accelerations are related $\alpha * r = a$.

Only the friction force produces a torque $\tau = r * F_f$.

So finally we have the following system of equations $\begin{cases} F_g^{\parallel} - F_f = m\,a \\ F_g^{\perp} = F_n \\ \tau = r\,F_f = I\,\alpha \end{cases}$

Substituting $\alpha * r = a$ in the above equation gives

$$\begin{cases} F_g^{\parallel} - m\,a = F_f \\ r\,F_f = I\,\dfrac{a}{r} \end{cases}$$

Combining both equations give

$$r^2 F_g^{\parallel} - r^2 m\,a = I\,a$$

Solving this for the linear acceleration a gives

$$a = \frac{r^2 F_g^{\parallel}}{I + m\,r^2}$$

As the moment of inertia of a cylinder along the rolling axis is $I = \frac{1}{2} m\,r^2$, and we have $F_g^{\parallel} = \sin(\beta) \times m \times g$, thus we finally have the linear acceleration of the rolling cylinder $a = \frac{2}{3} \sin(\beta) g$.